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# $c = 1$ String as the Topological Theory of the Conifold

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We show that the non-critical  $c = 1$  string at the self-dual radius is equivalent to topological strings based on the deformation of the conifold singularity of Calabi-Yau threefolds. The Penner sum giving the genus expansion of the free energy of the  $c = 1$  string theory at the self-dual radius therefore gives the universal behaviour of the topological partition function of a Calabi-Yau threefold near a conifold point.

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String theories compactified on Calabi-Yau manifolds are well motivated. These theories have also been extensively studied and we have been rewarded by some rather novel and unexpected features, which, it is believed, will enrich our understanding of the non-perturbative features of string theory. Particularly intriguing is the occurrence of a generic class of singularities, called conifolds [1], in the moduli spaces of Calabi-Yau threefolds. At these points, the threefold develops a singularity where a 3-cycle shrinks to zero size. This would lead to some apparently singular physical behaviour. A beautiful resolution to this singularity problem (at least for the type II strings) has recently been proposed in [2], where it is shown that the problem has its origin in our understanding of the low energy effective field theory near the conifold. Black holes that were previously massive, become light and eventually massless as we hit the singularity. This was used to propose [3] a smooth non-perturbative stringy process interpolating two completely different Calabi-Yau threefolds thereby changing the topology of the internal space. A test of this proposal was made in [4] by studying the topological partition function of the Calabi-Yau near the conifold at one-loop. It was observed there that there should be some universal features of the topological partition function at one loop as a consequence of the proposal in [2]. This was verified in some examples for which the topological partition function at one loop was computed.

In this paper we will demonstrate a further generalization of this universality by arguing that the  $c = 1$  string theory at the self-dual radius is equivalent to the topological theory (of the B-model) corresponding to the deformation near a conifold. A consequence of this fact is that the genus expansion of the free energy of the solvable  $c = 1$  string theory (at the self-dual radius) captures the universal singular behaviour of the free energy of topological partition function of threefolds near a conifold point. We will illustrate this by the example of the degenerating quintic. Fortunately, our task is greatly simplified by the fact that most of the relevant observations have been made in the literature. In the following, we will recall the necessary facts and show how they lead to the proposed link.

Recall that a conifold is a singular three (complex) dimensional Calabi-Yau manifold. By varying the moduli that deform the complex structure of the threefold, one can, by a finite change of the complex structure, reach a Calabi-Yau space which has a singularity [1]. Near the singularity, the local structure of the degenerating threefold can be described by the quadric [1]:

$$x^2 + y^2 + z^2 + t^2 = \mu$$

where  $x, y, z, t$  are coordinates of  $\mathbf{C}^4$ . As the parameter  $\mu \rightarrow 0$  the quadric develops a conical singularity at the origin. It is convenient for our purpose to make a linear redefinition of the local coordinates so that the quadric takes the following form

$$x_1x_2 - x_3x_4 = \mu \tag{1}$$

This change of variable is standard, and lets us infer that the base of the cone is  $S^3 \times S^2$ . Turning on a non-zero  $\mu$  amounts to smoothening the singularity by replacing the apex of the cone by an  $S^3$  of ‘size’  $\mu$ . (There is another way to desingularize the conifold, called the ‘small resolution’. In this process the origin is replaced by a  $\mathbf{CP}^1 \sim S^2$ . We will however focus our attention on the deformation by  $\mu$ .)

Let us now recall some aspects of the topological theory of Calabi-Yau threefolds coupled to topological gravity. In [5] it was shown that one can couple topological sigma models to topological gravity, and in many ways, such theories behave like ordinary non-critical matter coupled to 2d gravity. Moreover it was noted that the topological sigma models based on Calabi-Yau threefolds have a special property. For these theories, the partition function is non-vanishing in all loops without the need of any operator insertion. In this sense these are critical theories, and the critical value of the central charge  $\hat{c}$  of the topological algebra is  $\hat{c} = 3$ .

Topological sigma models on Calabi-Yau threefolds were studied in great detail in [6] with the discovery of the importance of holomorphic/topological anomalies in such cases. Moreover a field theory for the so called B-model of this topological theory was constructed. Since it describes the behaviour of the moduli that are sensitive to the deformation of the complex structure of the Calabi-Yau, it is called the ‘Kodaira-Spencer theory of gravity’. The Kähler moduli are frozen in the Kodaira-Spencer theory —holomorphic 3-form preserving diffeomorphisms of the threefold are the symmetries of the theory.

The topological algebra underlying any string background, in particular minimal matter coupled to gravity were found in [7][8]. The interesting case of  $c = 1$  theory, that is one free scalar compactified at the self-dual radius, coupled to gravity was explored in [9], where the topological  $SL(2, R)_3/U(1)$  coset model [10] was found to be equivalent to this background in the KPZ formalism. This work also provides an explanation for the observations of [11][10] concerning the relation between the  $c = 1$  string at self-dual radius to the Penner model. Moreover it was noted that although the theory is traditionally interpreted as a string moving in two dimensions, the topological theory is critical — it has the same central charge as that of a Calabi-Yau threefold!

The simplest way to describe this topological theory is in terms of a Landau-Ginzburg theory. The topological Landau-Ginzburg model in question is characterized by a singular superpotential  $W(X) = -\mu X^{-1}$  [12][13][14]. This model successfully reproduces the results of matrix model at tree level [13] and has an integrable structure underlying it [14]. Moreover it can also be used to compute the tachyon correlators at higher loops [15].

A step towards geometrization of this Landau-Ginzburg theory was taken in [16] by noting that one can relate it to a Calabi-Yau threefold by the usual trick of adding extra quadratic terms to the superpotential. Since the central charge is that of a threefold, (quadratic terms do not contribute to the central charge), one needs four extra fields, which together with the original variable  $x$  make up for the coordinates of the ambient space. The superpotential now takes the form

$$W(x) = -\mu x^{-1} + y_1^2 + y_2^2 + y_3^2 + y_4^2$$

For this equation to make sense the space in which it is embedded should be a weighted projective space  $\mathbf{WCP}_{-2,1,1,1,1}^4$ . Notice that since the degree of  $W$  equals the sum of weights, the manifold defined by  $W = 0$  satisfies the Calabi-Yau condition. It turns out to be convenient for us to represent the modified superpotential in an equivalent way by redefining variables

$$W = -\mu x^{-1} + x_1 x_2 - x_3 x_4 \tag{2}$$

Viewed in this geometric form, the locus  $W = 0$  is nothing but the conifold! More precisely the Calabi-Yau phase of the  $c = 1$  string theory is the deformation of a threefold near a conifold singularity. The quick way to see this is to go to the coordinate patch where  $x \neq 0$ , choose  $x = 1$  without any loss of generality, and note that in this affine patch  $W = 0$  is equivalent to the definition of the deformed conifold. The cosmological constant  $\mu$  in Eq.(1) plays the role of the complex modulus whose vanishing signals the appearance of the conifold singularity. Note that the coordinate patch with  $x = 0$  has an infinite potential associated with it and thus it does not arise in the geometrical description of this theory. This correspondence can be put on a more rigorous footing using the gauged linear sigma model approach [17].

Putting all this together we can conclude that the topological theory (in the B-model) describing the conifold coupled to topological gravity is equivalent to the  $c = 1$  string theory at the self-dual radius. Since we have used ingredients from different sources in reaching this conclusion, it would be desirable to have a more direct check of these ideas.

In the following, we will sketch two independent arguments which further support the above reasoning.

The first is the observation by Witten [18] (see also [19]) that various facts about the  $c = 1$  theory at the self-dual radius with  $\mu = 0$  can be best described by studying the geometry of the so called ground ring manifold  $Q$  given by

$$x_1x_2 - x_3x_4 = 0$$

The quadric cone  $Q$  is precisely the conifold, provided we interpret the variables  $x_i$  as *complex* coordinates<sup>4</sup>. This gives us an understanding of the main result of [18][19] which is the identification of physical states of  $c = 1$  string theory with various cohomology elements of  $Q$ . Indeed in a topological theory the BRST states correspond to the cohomology of the manifold on which the sigma model lives. In this identification, the ghost numbers of the left- and right-moving components of the physical states are, up to shifts, the (anti-)holomorphic degrees of the differential forms on  $Q$ .

Moreover the symmetries of this string theory are the ‘volume preserving diffeomorphisms’ of the quadric cone  $Q$ . This was really established for the case of vanishing cosmological constant. However, it was argued that the conclusions remain unchanged for non-zero  $\mu$ , which has the effect of smoothening the singularity (see [21] for support of this argument). Now, in view of our interpretation, the symmetry should be elevated to the ‘holomorphic three-form preserving’ analytic diffeomorphisms of  $Q$ . In fact this was one of the hints provided in [6] for a connection between topological string theory on Calabi-Yau manifolds and the  $c = 1$  string.

Compared to the compact Calabi-Yau manifolds, there is however an important difference in this case. The threefold here is non-compact and since we have not specified the boundary conditions the dimension of the cohomology can be infinite. This is the reason for having infinitely many tachyons and other discrete states in the  $c = 1$  string theory.

A second check follows from the observation made in [6] concerning the behaviour of the singularity of partition functions of Calabi-Yau manifolds near the conifold points. The nature of this singularity is identical to that of the  $c = 1$  string at all loops in the genus expansion of the free energy. It was suggested that in this sense, topological strings based on Calabi-Yau threefolds are in the same universality class as the  $c = 1$  string.

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<sup>4</sup> This correspondence immediately suggests a generalization of the topological model to  $c = 1$  string at other special radii of compactification[20].

However if our identification is correct not only should the singularity structure be the same but also the coefficients multiplying the singularity in the genus expansion of the partition function must match exactly. This is expected since the free energy near the conifold point is dominated by the development of the singularity.

The  $c = 1$  string theory at the self-dual radius have been solved in many different ways. For this theory, we know the exact expression of the free energy as a function of the cosmological constant  $\mu$ . This is given as a sum over world-sheet of different genera [22][11][9][15]:

$$F = \frac{1}{2}\mu^2 \log \mu - \frac{1}{12} \log \mu + \frac{1}{240}\mu^{-2} + \sum_{g>2} a_g \mu^{2-2g} \quad (3)$$

where the coefficient  $a_g = B_{2g}/2g(2g-2)$  is the Euler class of the moduli space of Riemann surfaces of genus  $g$ .

Let us start with genus zero. This is indeed the well-known universal behavior for the conifold. When  $\mu$  parametrizes the period of the vanishing three cycle, there exists, in the symplectic basis, a conjugate cycle whose period is proportional to  $\mu \log \mu$ . It follows from special geometry that the free energy is given by its integral[1][23]. This of course agrees with the tree level contribution to the free energy of the  $c = 1$  string. It also helps us identify the normalization of  $\mu$  against the complex modulus of the degenerating threefold.

Let us now go to the one loop result. In that case, as has recently been discussed in [4], there is a lot of evidence that the coefficient of  $-\frac{1}{12}$  accompanying  $\log \mu$  is the universal behaviour of the one loop topological partition function of the conifold. This would in fact be necessary for the consistency of the physical resolution the conifold singularity in string theory as proposed in [2][3].

This brings us to the higher loop coefficients, which would certainly be a rather non-trivial check of our proposed link between the  $c = 1$  string at self-dual radius and topological theory of the conifold. Here we notice that even though all the higher genus free energy of the topological sigma models based on Calabi-Yau spaces have not yet been computed, the genus-2 behaviour of the celebrated quintic threefold has been studied in detail in [6].

We should start by fixing the normalization. In the notation of [6], the complex modulus  $\psi$  parametrizes the behaviour of the quintic near the conifold point  $\psi = 1$ . The relation between  $\mu$  and  $\psi$  is determined by the genus zero term as

$$\mu \sim \left( \frac{5}{2\pi i} \right)^3 (1 - \psi)$$

In [6], the partition function depends on  $\mu$  and  $\bar{\mu}$ , (that is why there is an ‘anomaly’ in such theories). Setting  $\bar{\mu} = 0$ , and expanding the free energy as a function of  $\mu$ , one can check that the anomaly contribution captured by the Feynman graphs of [6] does not affect the leading singularity, which entirely comes from the coefficient  $C$  (see p. 398–399 of [6])<sup>5</sup>. Using the normalization fixed as above, the coefficient of the  $\mu^{-2}$  term is  $\frac{1}{240}$ , exactly as in the  $c = 1$  theory!

It would be desirable to make an explicit check of the other higher loop coefficients. However, on the basis of our general reasoning, we expect them to match exactly; in fact turning this around, the above result provides one more data to fix the holomorphic piece [6] of the topological partition functions at higher genera!

Thus we come to the remarkable conclusion that the genus expansion of the free energy of the  $c = 1$  string theory at the self-dual radius gives the universal singular behaviour of the free energy of an arbitrary topological sigma model on a Calabi-Yau threefold near a conifold point of its moduli space.

This link between the conifold and the  $c = 1$  string theory is sure to have many implications. The latter is particularly well understood from many different points of view (for a review see [24]). Once the full implication of the correspondence is clear, it might help understand the structure of the conifold transition which would be important in further developing the recent results of [2]. If so, just as with the partition function discussed in [4], for each fixed genus the tachyon scattering amplitudes in this theory will have a bearing as exact computations for the string theory near the conifold and should be further reviewed in this light. Note also that on the other hand, as far as the  $c = 1$  theory is concerned, the string field theory is thus the Kodaira-Spencer theory of [6] expanded in the conifold background, and should provide an alternative method for computing the scattering amplitudes in this theory<sup>6</sup>.

In [2][3], black holes that are solitonic solutions of the string theory play a crucial role. The singularity in the effective field theory near a conifold is caused by integrating them out. Since the  $c = 1$  theory captures precisely this singularity, there is probably a more direct connection of the modes connected with these light black holes and the excitations

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<sup>5</sup> There is an implicit  $1/(2\pi i)^3$  in the definition of  $C$  as given in in [6]. This arises because in defining the Yukawa couplings on sphere the theta angle was chosen to be periodic with period one rather than  $2\pi$ .

<sup>6</sup> Geometrically the tachyons are  $(2, 1)$  forms on the threefold.

of the former. In this context, it is intriguing to note that there are black hole solutions in the  $c = 1$  string theory[25][9].

The lesson we are learning is that we should identify the singularities of Calabi-Yau manifolds with scaling limits of the  $c = 1$  string. However there are many types of singularities of Calabi-Yau manifolds, and for each one of them we may get a different universal theory. Here we have only identified the one corresponding to the simple conifold singularity. One might also be tempted to identify the  $c = 1$  string at other special radii with degenerations of Calabi-Yau manifold. For example at  $n$  times the self-dual radius at zero cosmological constant the expected ring is  $(xy)^n = zt$  [26]. This does describe a three dimensional non-compact singular manifold and one may wonder whether it can arise as a part of a compact Calabi-Yau manifold. It seems unlikely that this particular type of singularity will appear as a part of a compact Calabi-Yau manifold for the following reason. The one loop partition function of this theory  $F_1 = -\frac{1}{24} (n + \frac{1}{n}) \log \mu$ , is not, (except for  $n = 1$ ), an integer multiple of  $-\frac{1}{12} \log \mu$  as is required for the resolution of the singularity[4].

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